First order structures

Suppose $\mathcal{L}$ is a first order language with constant symbols $\{c_i\}_{i\in I}$, function symbols $\{f_j\}_{j\in J}$ and relation symbols $\{R_k\}_{k\in K}$.

An <u>$\mathcal{L}$-structure</u> $\mathcal{M} = \langle M, \{c_i^{\mathcal{M}}\}_{i\in I}, \{f_j^{\mathcal{M}}\}_{j\in J}, \{R_k^{\mathcal{M}}\}_{k\in K}\rangle$ is a tuple consisting of

(i) a non-empty set M, called the <u>universe</u> of $\mathcal{M}$,

(ii) for every $i \in I$, an element $c_i^{\mathcal{M}} \in M$,

(iii) for every $j \in J$, if f_j is an n-ary function symbol,

$$f_j^{\mathcal{M}} : M^n \longrightarrow M$$

is a function

(iv) for every $k \in K$, if R_k is an n-ary relation symbol, $R_k^{\mathcal{M}} \subseteq M^n$ is a subset.

In this case, $c_i^{\mathcal{M}}$, $f_j^{\mathcal{M}}$ and $R_k^{\mathcal{M}}$ are called the <u>interpretations</u> of the symbols of L in the structure $\mathcal{M}$.

<u>Example</u> Suppose $L = \{c, f, g\}$, where c is a constant and f, g are unary and binary fct. symbols resp.

Let $\mathcal{M}$ be the structure with universe $\mathbb{Z}$ and where $c^{\mathcal{M}} = 1$, $f^{\mathcal{M}}(n) = -n$, and $g^{\mathcal{M}}(n, m) = n + m$. Then

$$\mathcal{M} = \langle \mathbb{Z}, c^{\mathcal{M}}, f^{\mathcal{M}}, g^{\mathcal{M}} \rangle$$

is an L-structure.

<u>Interpretation of terms</u>

A term is said to be <u>variable free</u> if it contains no occurrences of any variables.

Then if $\mathcal{M}$ is an $\mathcal{L}$-structure, we can extend the interpretations of constants to all variable free terms of L:

Definition:

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and s a variable free L-term. The interpretation $s^{\mathcal{M}} \in M$ is defined by induction on s:

- If s is a constant symbol c, set
  $$s^{\mathcal{M}} = c^{\mathcal{M}},$$
- If $s = f(t_1, t_2, \ldots, t_n)$, where f is a function symbol and $t_1, \ldots, t_n$ are terms, set
  $$s^{\mathcal{M}} = f^{\mathcal{M}}(t_1^{\mathcal{M}}, t_2^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}}).$$

Example

Consider the structure $\mathcal{M}$ above. Then

$$s = f(g(c, f(c)))$$

is a variable free L-term.

$$s^{\mathcal{M}} = f^{\mathcal{M}}(g^{\mathcal{M}}(c^{\mathcal{M}}, f^{\mathcal{M}}(c^{\mathcal{M}})))$$

$$= -(1 + (-1)) = -0 = 0.$$

So variable free terms can be seen as names for objects in M.

Free and bound variables

If A is an L-formula and v a variable, then each occurrence of v in A is either <u>bound</u> or <u>free</u>.

E.g., in

$$\forall \underline{x_7} \; \left(R\underline{x_7}x_2 \;\vee\; x_2 = x_3 \right) \longrightarrow \exists \underline{x_2} \; \underline{x_2} = f(x_7, \underline{x_2})$$

the underlined occurrences of variables are bound while all other are free.

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Truth in a structure

Suppose A is an $\mathcal{L}$-formula with no free variables and $\mathcal{M}$ is an $\mathcal{L}$-structure.

We define <u>satisfaction</u> of A or <u>truth</u> of A in this structure $\mathcal{M}$ by induction on A as follows:

- If A is the formula $t = s$, where t, s are variable free $\mathcal{L}$-terms,

$$\mathcal{M} \models A \iff t^{\mathcal{M}} = s^{\mathcal{M}}$$

Naming elements in a structure.

Suppose $\mathcal{M}$ is an L-structure and let $L_{\mathcal{M}}$ be the expanded language in which for every $a \in M$ we have added a new constant i_a to the language.

We can then see $\mathcal{M}$ as an $L_{\mathcal{M}}$ structure by defining $(i_a)^{\mathcal{M}} = a \in M$.

This corresponds to <u>naming</u> all the elements of $\mathcal{M}$.

Substitution of terms for variables in a formula

Suppose $v_1, \ldots, v_k$ are distinct variables, $t_1, \ldots, t_n$ are L-terms and A is an L-formula, we define the <u>simultaneous</u> <u>substitution</u> of $t_1, \ldots, t_n$ for the <u>free</u> <u>occurrences</u> of $v_1, \ldots, v_n$ in A by induction on A:

- If A is the formula $s = r$, where s, r are terms, then

$$A[t_1/v_1, \dots, t_n/v_n]$$

is the formula

$$s[t_1/v_1, \dots, t_n/v_n] = r[t_1/v_1, \dots, t_n/v_n]$$

- If $A = Rs_1 \dots s_m$,

$$A[t_1/v_1, \dots, t_n/v_n] = Rs_1[t_1/v_1, \dots, t_n/v_n] \dots s_m[t_1/v_1, \dots, t_n/v_n]$$

- If $A = QxB$, where Q is a quantifier and $x \neq v_1, \dots, v_n$, then

$$A[t_1/v_1, \dots, t_n/v_n] = QxB[t_1/v_1, \dots, t_n/v_n]$$

- If $A = Qv_iB$, ~~then~~

$$A[t_1/v_1, \dots, t_n/v_n] = Qv_i\, B[t_1/v_1, \dots, t_{i-1}/v_{i-1}, t_{i+1}/v_{i+1}, \dots, t_n/v_n]$$

- The cases of $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ are trivial.

Example

$$A = (\forall x \exists z\, Rxz) \wedge \exists y\; x = y$$

Then $A[s/x, t/y] = (\forall x \exists z\, Rxz) \wedge \exists y\; s = y$.

- If A is the formula $R t_1 t_2 \ldots t_n$, where R is an n-ary relation symbol and $t_1, \ldots, t_n$ are variable free L-terms,

$$\mathcal{M} \models A \iff (t_1^{\mathcal{M}}, t_2^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}}) \in R^{\mathcal{M}}.$$

- If A and B are L-formulas with no free variables, set

$$\mathcal{M} \models \neg A \iff \mathcal{M} \not\models A$$

$$\mathcal{M} \models (A \wedge B) \iff \mathcal{M} \models A \text{ \underline{and} } \mathcal{M} \models B$$

$$\mathcal{M} \models (A \vee B) \iff \mathcal{M} \models A \text{ \underline{or} } \mathcal{M} \models B$$

$$\mathcal{M} \models (A \to B) \iff \text{if } \mathcal{M} \models A, \text{ then } \mathcal{M} \models B$$

$$\mathcal{M} \models (A \leftrightarrow B) \iff \mathcal{M} \models A \text{ if and only if } \mathcal{M} \models B.$$

- If A is an L-formula with at most one free variable v, set

$$\mathcal{M} \models \forall v\, A \iff \text{for any } a \in M$$

$$\mathcal{M} \models A[{}^{i_a}/_{v}]$$

and

$$\mathcal{M} \models \exists v\, A \iff \text{for some } a \in M \quad \mathcal{M} \models A[{}^{v_a}/_v].$$

<u>Example</u>

Consider the language $L = \{c, f, R\}$,

where

- c is a constant
- f is a unary function symbol
- R is a binary relation symbol.

Let A be the formula,

$$\forall x \forall y \left(Rxy \longrightarrow \exists z (Rxz \wedge Rzy) \right),$$

B ~~the~~ formula,

$$\forall x \forall y \left(x = c \vee Rf(x)x \right).$$

Define also structures $\mathcal{M}$, $\mathcal{N}$, $\mathcal{L}$ by

$\mathcal{M} = \langle \mathbb{Z}, 0, x-1, < \rangle$,

i.e., $c^{\mathcal{M}} = 0$, $f^{\mathcal{M}}(x) = x-1$, $R^{\mathcal{M}} = <$.

$\mathcal{N} = \langle \mathbb{Q}, 0, x-1, < \rangle$

$\mathcal{L} = \langle \mathbb{N}, 0, x \dot{-} 1, < \rangle$

where $x \dot{-} 1 = \begin{cases} x-1 & \text{if } x \geqslant 1 \\ x & \text{if } x = 0 \end{cases}$.

Then

$$\mathcal{M} \models \forall x \forall y \left(Rxy \rightarrow \exists z \left(Rxz \wedge Rzy \right) \right)$$

$\iff$ for all $m, n \in \mathbb{Z}$,

if $\mathcal{M} \models R i_m i_n$, then there is some $k \in \mathbb{Z}$

s.t. $\mathcal{M} \models R i_m i_k \wedge R i_k i_n$

$\iff$ for all $m, n \in \mathbb{Z}$,

if $m < n$, then there is some $k \in \mathbb{Z}$ s.t.

$m < k$ and $k < n$.

Since this latter statement is false (take $m = 0$, $n = 1$)

we see that

$$\mathcal{M} \not\models \forall x \forall y \left(Rxy \to \exists z (Rxz \wedge Rzy) \right).$$

On the other hand,

$$\mathcal{N} \models \forall x \forall y \left(Rxy \longrightarrow \exists z (Rxz \wedge Rzy) \right)$$

and

$$\mathcal{L} \models \forall x \forall y \left(x = c \vee Rf(x)x \right).$$

<u>Definition</u>

Suppose A is a formula of a first order language L whose free variables are among $v_1, \ldots, v_n$. Then if $\mathcal{M}$ is an L-structure, we say that A is <u>true</u> in $\mathcal{M}$, $\mathcal{M} \models A$, if

$$\mathcal{M} \models \forall v_1 \forall v_2 \ldots \forall v_n \, A.$$

Example

The formula $x = y \vee x \neq y$ is true in any structure

$$\mathcal{M} \models x = y \vee x \neq y$$

since

$$\mathcal{M} \models \forall x \forall y \, (x = y \vee x \neq y).$$

Models of a theory

Suppose T is a theory of a first order language L, i.e., T is a collection of L-formulas. An L-structure $\mathcal{M}$ is said to be a <u>model</u> of T, $\mathcal{M} \models T$, if

$$\mathcal{M} \models B \quad \text{for any } B \in T.$$

Also, if A is any L-formula, we say that A is <u>valid</u> in T, $T \models A$, if

$$\mathcal{M} \models A \quad \text{for any } \mathcal{M} \models T.$$

So $T \models A$ if A is true in any model of T.

<u>Example</u>

The language of fields has two constant symbols $0, 1$ and two binary function symbols $\cdot, +$.

The theory of fields has axioms

Theory of Abelian gps

1: $\forall x \forall y \forall z \quad (x+y)+z = x+(y+z)$

2: $\forall x \quad x+0=x$

3: $\forall x \exists y \quad x+y=0$

4: $\forall x \forall y \quad x+y=y+x$

Commutative ring with identity

5: $\forall x \forall y \forall z \quad (x \cdot y)\cdot z = x \cdot (y \cdot z)$

6: $\forall x \quad x \cdot 1 = x$

7: $\forall x \forall y \forall z \quad x \cdot (y+z) = (x \cdot y)+(x \cdot z)$

8. $\forall x \forall y \quad x \cdot y = y \cdot x$

9: $\forall x \left(\neg x=0 \longrightarrow \exists y \; x \cdot y = 1 \right)$

10: $0 \neq 1$.

Recall that the characteristic of a field $\mathbb{F}$

is 0 if $\underbrace{1+1+\dots+1}_{n \text{ times}} \neq 0$ for all $n \geq 2$,

if not, the characteristic of $\mathbb{F}$ is the smallest $n \geq 2$ such that

$$\underbrace{1+1+\dots+1}_{n \text{ times}} = 0 .$$

If the characteristic of $\mathbb{F}$ is $\neq 0$, it must be prime.

Write T_{fields} for the theory of fields.

So if $\mathbb{Q} = \langle \mathbb{Q}, 0, 1, +, \cdot \rangle$ is the field of rational numbers, we see that

$$\mathbb{Q} \models \forall x \ \neg (x \cdot x = 1+1) .$$

I.e., the polynomial $x^2 = 2$ has no roots in $\mathbb{Q}$.

Similarly, let $\mathcal{R} = \langle \mathbb{R}, 0, 1, +, \cdot \rangle$ and $\mathcal{C} = \langle \mathbb{C}, 0, 1, +, \cdot \rangle$.

Then $\mathcal{R} \not\models \forall x \, \neg (x \cdot x = 1 \cdot 1)$

and $\mathcal{C} \not\models \forall x \, \neg (x \cdot x = 1 \cdot 1)$.

A field $\mathbb{F}$ is said to be <u>algebraically closed</u> if any non-constant polynomial with coefficients in $\mathbb{F}$ has a root in $\mathbb{F}$.

ACF is the theory of algebraically closed fields, that is T_{fields} + the following axioms

$$\forall x_1 \forall x_2 \dots \forall x_n \exists y \quad y^n + x_n y^{n-1} + x_{n-1} y^{n-2} + \dots + x_1 = 0$$

for every n. Here y^n is shorthand for

$$\underbrace{(\dots((y \cdot y) \cdot y) \dots \cdot y)}_{n \text{ times}}$$

A class of L-structures $\mathcal{C}$ is said to be <u>axiomatisable</u> if there is an L-theory T such that

$$\mathcal{C} = \{ \mathcal{M} \text{ an } L\text{-structure} \mid \mathcal{M} \models T \}.$$

So, for example, the class $\mathcal{C}$ of algebraically closed fields is axiomatisable, since

$$\mathcal{C} = \{ \mathcal{M} \mid \mathcal{M} \models ACF \}.$$

<u>Exercise</u>

Show that the class of algebraically closed fields of characteristic 0 is axiomatisable. Give the corresponding theory ACF_0.

Isomorphism

Suppose L is a language with constant symbols $\{c_i\}_{i\in I}$, function symbols $\{f_j\}_{j\in J}$ and relation symbols $\{R_k\}_{k\in K}$.

Let $\mathcal{M} = \langle M, \{c_i^{\mathcal{M}}\}_{i\in I}, \{f_j^{\mathcal{M}}\}_{j\in J}, \{R_k^{\mathcal{M}}\}_{k\in K}\rangle$ and $\mathcal{N} = \langle N, \{c_i^{\mathcal{N}}\}_{i\in I}, \{f_j^{\mathcal{N}}\}_{j\in J}, \{R_k^{\mathcal{N}}\}_{k\in K}\rangle$ be L-structures.

An <u>embedding</u> $\phi : \mathcal{M} \longrightarrow \mathcal{N}$ of $\mathcal{M}$ into $\mathcal{N}$ is a function $\phi : M \longrightarrow N$ such that

(i) ϕ is injective, i.e., one-to-one.

(ii) for any $i \in I$, $\phi(c_i^{\mathcal{M}}) = c_i^{\mathcal{N}}$

(iii) for any $j \in J$, if f_j is an n-ary function symbol and $a_1, \ldots, a_n \in M$, then

$$\phi\left(f_j^{\mathcal{M}}(a_1, \ldots, a_n)\right) = f_j^{\mathcal{N}}\left(\phi(a_1), \phi(a_2), \ldots, \phi(a_n)\right)$$

(iv) for any $k \in K$, if R_k is an n-ary relation symbol and $a_1, \ldots, a_n \in M$, then

$$(a_1, \ldots, a_n) \in R_k^{\mathcal{M}} \iff (\phi(a_1), \ldots, \phi(a_n)) \in R_k^{\mathcal{N}}.$$

<u>Example</u>

Let $L = \{e, \cdot\}$ be the language of groups with a constant symbol e for the identity element and a binary function symbol $\cdot$ for the group multiplication.

T = the theory of groups

$G = \langle \mathbb{R}, 0, + \rangle$ = additive group of reals

$H = \langle \mathbb{R}_+, 1, \cdot \rangle$ = group of positive real numbers under multiplication.

Then $\exp : G \longrightarrow H$ given by

$\exp(x) = e^x$ is an embedding of G into H:

$$\exp(x \cdot^G y) = \exp(x+y) = e^{x+y}$$
$$= e^x e^y = \exp(x) \cdot^H \exp(y). \quad \exp(0) = 1.$$

<u>Example</u> Let $L = \{0, +, <\}$ be the language of ordered abelian groups and let T_{OAG} be the corresponding theory having axioms

- $\forall x \forall y \quad x+y = y+x$
- $\forall x \forall y \forall z \quad x+(y+z) = (x+y)+z$
- $\forall x \exists y \quad x+y = 0$
- $\forall x \quad x = x+0$
- $\forall x \forall y \forall z \quad (x<y \longleftrightarrow x+z < y+z)$.

Then $G = \langle \mathbb{Z}, 0, +, < \rangle$ and $H = \langle \mathbb{Q}, 0, +, < \rangle$ are both models of T_{OAG} and the map

$$\phi : \mathbb{Z} \to \mathbb{Q} \quad , \quad \phi(m) = \frac{m}{7}$$

is an embedding.

<u>Question</u> Is $\psi : \mathbb{Z} \to \mathbb{Q}$, $\psi(m) = -m$ an embedding?

<u>Definition</u> An <u>isomorphism</u> is a surjective embedding.

Thus, for example, the map

$$\exp : \langle \mathbb{R}, 0, + \rangle \longrightarrow \langle \mathbb{R}_+, 1, \cdot \rangle$$

is an isomorphism with inverse log.

<u>Lemma</u> Suppose $\mathcal{M}$ and $\mathcal{N}$ are L-structures and $\phi : \mathcal{M} \longrightarrow \mathcal{N}$ is an embedding. Then if t is an L-term whose free variables are among $v_1, \ldots, v_n$, we have for any $a_1, \ldots, a_n \in M$

$$\phi\left(t\left[{}^{i_{a_1}}/_{v_1}, \ldots, {}^{i_{a_n}}/_{v_n} \right]^{\mathcal{M}} \right) = t\left[{}^{i_{\phi(a_1)}}/_{v_1}, \ldots, {}^{i_{\phi(a_n)}}/_{v_n} \right]^{\mathcal{N}}$$

For simplicity we will write $t[v_1, \ldots, v_n]$ to indicate that all free variables of t are among $v_1, \ldots, v_n$.

Also, instead of $t[{}^{i_{a_1}}/{}_{v_1}, \dots, {}^{i_{a_n}}/{}_{v_n}]$ we will write $t[i_{a_1}, \dots, i_{a_n}]$ or even $t[i_{\vec{a}}]$

Similarly, $\phi(\vec{a}) := (a_1, a_2, \dots, a_n)$.

Proof By induction on the construction of t.

- t a variable v_i:

$$\phi(v_i[i_{\vec{a}}]^{\mathcal{M}}) = \phi(i_{a_i}^{\mathcal{M}}) = \phi(a_i)$$
$$= i_{\phi(a_i)}^{\mathcal{N}} = v_i[i_{\phi(\vec{a})}]^{\mathcal{N}}.$$

- t a constant symbol c:

$$\phi(c[i_{\vec{a}}]^{\mathcal{M}}) = \phi(c^{\mathcal{M}}) = c^{\mathcal{N}} = c[i_{\phi(\vec{a})}]^{\mathcal{N}}$$

- t the term $f(t_1, \dots, t_m)$:

$$\phi(f(t_1, \dots, t_m)[i_{\vec{a}}]^{\mathcal{M}}) = \phi(f(t_1[i_{\vec{a}}], \dots, t_m[i_{\vec{a}}])^{\mathcal{M}})$$
$$= \phi(f^{\mathcal{M}}(t_1[i_{\vec{a}}]^{\mathcal{M}}, \dots, t_m[i_{\vec{a}}]^{\mathcal{M}}))$$
$$= f^{\mathcal{N}}(\phi(t_1[i_{\vec{a}}]^{\mathcal{M}}), \dots, \phi(t_m[i_{\vec{a}}]^{\mathcal{M}}))$$
$$= f^{\mathcal{N}}(t_1[i_{\phi(\vec{a})}]^{\mathcal{N}}, \dots, t_m[i_{\phi(\vec{a})}]^{\mathcal{N}})$$
$$= f(t_1[i_{\phi(\vec{a})}], \dots, t_m[i_{\phi(\vec{a})}])^{\mathcal{N}}$$
$$= f(t_1, \dots, t_m)[i_{\phi(\vec{a})}]^{\mathcal{N}}.$$

$\square$

To further simplify notation, when $t[v_1, \dots, v_n]$ is a term and $A[v_1, \dots, v_n]$ a formula, write $t[a_1, \dots, a_n]$ and $A[a_1, \dots, a_n]$ for $t[\bar{i\bar{a}}/\bar{v}]$, resp. $A[\bar{i\bar{a}}/\bar{v}]$ .20

Prop Suppose $\phi : \mathcal{M} \to \mathcal{N}$ is an embedding of L-structures. Let $A[v_1, \dots, v_n]$ be a quantifier free formula and let $a_1, \dots, a_n$ be elements of M. Then

$$\mathcal{M} \models A[\bar{a}] \iff \mathcal{N} \models A[\phi(\bar{a})].$$

Proof By the previous lemma, we have for any term $t[v_1, \dots, v_n]$,

$$\phi\left(t[a_1, \dots, a_n]^{\mathcal{M}}\right) = t[\phi(a_1), \dots, \phi(a_n)]^{\mathcal{N}}.$$

We prove the result by induction on the construction of A.

Suppose first A is the formula $t = s$ for terms $t[v_1, \dots, v_n]$ and $s[v_1, \dots, v_n]$.

Then $$\mathcal{M} \models A[\bar{a}] \iff t[\bar{a}]^{\mathcal{M}} = s[\bar{a}]^{\mathcal{M}}$$
$$\iff \phi\left(t[\bar{a}]^{\mathcal{M}}\right) = \phi\left(s[\bar{a}]^{\mathcal{M}}\right)$$

$$\iff t[\phi(\bar{a})]^{\mathcal{N}} = s[\phi(\bar{a})]^{\mathcal{N}}$$

$$\iff \mathcal{N} \models A[\phi(\bar{a})]$$

And if A is the formula $Rt_1 \cdots t_m$, where R is an m-ary relation symbol and $t_1[\bar{v}], \ldots, t_m[\bar{v}]$ are terms, then

$$\mathcal{M} \models A[\bar{a}] \iff \left(t_1[\bar{a}]^{\mathcal{M}}, \ldots, t_m[\bar{a}]^{\mathcal{M}}\right) \in R^{\mathcal{M}}$$

$$\iff \left(\phi\left(t_1[\bar{a}]^{\mathcal{M}}\right), \ldots, \phi\left(t_m[\bar{a}]^{\mathcal{M}}\right)\right) \in R^{\mathcal{N}}$$

$$\iff \left(t_1[\phi(\bar{a})]^{\mathcal{N}}, \ldots, t_m[\phi(\bar{a})]^{\mathcal{N}}\right) \in R^{\mathcal{N}}$$

$$\iff \mathcal{N} \models A[\phi(\bar{a})]$$

This shows that the proposition holds for atomic formulas. Since it is easy to see that it also holds for any boolean combination of atomic formulas, this verifies the proposition. $\square$

<u>Theorem</u> Let $\phi : \mathcal{M} \to \mathcal{N}$ be an isomorphism of L-structures. Then for any formula $A[v_1, \dots, v_n]$ and $a_1, \dots, a_n \in M$,

$$\mathcal{M} \models A[\bar{a}] \iff \mathcal{N} \models A[\phi(\bar{a})].$$

<u>Examples</u>

Consider the language $L = \{e, \otimes\}$ of group theory and let

$$\mathcal{Z} = \langle \mathbb{Z}, 0, + \rangle, \quad \mathcal{R} = \langle \mathbb{R}_+, 1, \cdot \rangle$$

be L-structures.

The map $\phi : \mathcal{Z} \to \mathcal{R}$, $\phi(a) = \exp(a)$, is an embedding of $\mathcal{Z}$ into $\mathcal{R}$.

So for any quantifier free L-formula $A[v_1, \dots, v_n]$ and any $a_1, \dots, a_n \in \mathbb{Z}$,

$$\mathcal{Z} \models A[a_1, \dots, a_n] \iff \mathcal{R} \models A[\phi(a_1), \dots, \phi(a_n)].$$

For example, let $A[x,y,z]$ be the formula

$$(x \otimes y) \otimes z = e \quad \wedge \quad x \otimes x = y$$

Then

$$\mathbb{Z} \models A[2,4,-6],$$

since $(2+4)+(-6) = 0 \quad \wedge \quad 2+2 = 4$.

Thus, $\mathcal{R} \models A[\phi(2), \phi(4), \phi(-6)]$,

which we can verify:

$$\frac{\exp(2) \cdot \exp(4)}{\exp(6)} = 1 \quad \wedge \quad \exp(2) \cdot \exp(2) = \exp(4).$$

However, $\mathcal{M}$ and $\mathcal{N}$ do not satisfy the same sentences (i.e., <u>closed</u> formulas or formulas w/o free variables. To see this, note that

$$\mathcal{R} \models \forall x \ (x \neq e \rightarrow \exists y \ y \otimes y = x)$$

while

$$\mathbb{Z} \not\models \forall x \ (x \neq e \rightarrow \exists y \ y \otimes y = x)$$

Exercise Why is the formula false in $\mathbb{Z}$?

Definition An embedding $\phi : \mathcal{M} \to \mathcal{N}$ between L-structures is said to be an elementary embedding if for any L-formula $A[v_1, \ldots, v_n]$ and $a_1, \ldots, a_n \in M$,

$$\mathcal{M} \models A[a_1, \ldots, a_n] \iff \mathcal{N} \models A[\phi(a_1), \ldots, \phi(a_n)].$$

Definition A substructure $\mathcal{N}$ of an L-structure $\mathcal{M} = \langle M, \ldots \rangle$ is an L-structure $\mathcal{N} = \langle N, \ldots \rangle$ where

(i) $N \subseteq M$

(ii) $c^{\mathcal{N}} = c^{\mathcal{M}} \in N$ for any constant $c \in L$

(iii) $R^{\mathcal{N}} = R^{\mathcal{M}} \cap N^n$ for any n-ary relation symbol $R \in L$

(iv) $f^{\mathcal{N}}(a_1, \ldots, a_n) = f^{\mathcal{M}}(a_1, \ldots, a_n) \in N$ for any $a_1, \ldots, a_n \in N$ and n-ary function symbol $f \in L$.

Remark If $\mathcal{N}$ is a substructure of $\mathcal{M}$, written $\mathcal{N} \subseteq \mathcal{M}$, then the identity map

$$\mathrm{id} : \mathcal{N} \to \mathcal{M}$$

is an embedding of $\mathcal{N}$ into $\mathcal{M}$.

Definition A substructure $\mathcal{N} \subseteq \mathcal{M}$ is said to be an elementary substructure of $\mathcal{M}$ if $\mathrm{id} : \mathcal{N} \to \mathcal{M}$ is an elementary embedding, i.e., if for any L-formula $A[v_1, \ldots, v_n]$ and $a_1, \ldots, a_n \in N$,

$$\mathcal{N} \models A[a_1, \ldots, a_n] \iff \mathcal{M} \models A[a_1, \ldots, a_n].$$

Note The parameters $a_1, \ldots, a_n$ always belong to N.

Examples

Consider $L = \{<\}$ and the following three L-structures:

$$\mathcal{Z} = \langle \mathbb{Z}, < \rangle$$
$$\mathcal{Q} = \langle \mathbb{Q}, < \rangle$$
$$\mathcal{R} = \langle \mathbb{R}, < \rangle.$$

Then $\mathbb{Z}$ is a substructure of $\mathbb{Q}$, which is again a substructure of $\mathbb{R}$.

$\mathbb{Z}$ is not an elementary substructure of $\mathbb{Q}$:

To see this, note that

$$\mathbb{Z} \models \forall x\, \big((x < 1 \wedge -1 < x) \longrightarrow x = 0\big)$$

while

$$\mathbb{Q} \not\models \forall x\, \big((x<1 \wedge -1<x) \longrightarrow x=0\big).$$

On the other hand, $\mathbb{Q}$ is an elementary substructure of $\mathbb{R}$, though we shall not prove this yet.

<u>Observation</u> If $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$, written $\mathcal{N} \preccurlyeq \mathcal{M}$, then $\mathcal{N}$ and $\mathcal{M}$ are elementary equivalent, $\mathcal{M} \equiv \mathcal{N}$.

<u>Example</u> Dense linear orderings w/o endpoints.

Let $L = \{<\}$ be the language of partial orderings and let T be the L-theory with axioms

Linear orderings
$$\begin{cases} \forall x \forall y \; (x<y \lor y<x \lor x=y) \\ \forall x \; \neg\, x<x \\ \forall x \forall y \forall z \; ((x<y \land y<z) \to x<z) \end{cases}$$

Dense without endpoints
$$\begin{cases} \forall x \forall y \; (x<y \to \exists z\, (x<z \land z<y)) \\ \forall x \exists y \exists z \; (z<x \land x<y) \end{cases}$$

L-structures that are models of T are said to be <u>dense</u> <u>linear</u> <u>orderings</u> <u>without</u> <u>endpoints</u>.

<u>Theorem</u> (Cantor) Any two countable dense linear orderings without endpoints are isomorphic

<u>Proof</u> The proof method is called back and forth.

Suppose $(A,<)$ and $(X,<)$ are ctbl. dense linear orderings w/o endpoints and enumerate the structures as

$$a_1, a_2, a_3, \ldots \quad \text{and} \quad x_1, x_2, x_3, \ldots \quad \text{resp.}$$

We will find a bijection $F: \mathbb{N} \to \mathbb{N}$ such that

$$a_n < a_m \iff x_{F(n)} < x_{F(m)} \qquad (*)$$

for all $n, m \in \mathbb{N}$.

In that case, letting $\phi(a_i) = x_{F(i)}$, we see that ϕ is an isomorphism of $(A,<)$ and $(X,<)$.

We begin by setting $F(1) = 1$.

Now, suppose $D \subseteq \mathbb{N}$ is a finite set and $F(d)$ is defined for all $d \in D$ such that $(*)$ holds for all $n, m \in D$.

Then let $D = \{d_1, \ldots, d_k\}$, where the enumeration is chosen such that

$$a_{d_1} < a_{d_2} < a_{d_3} < \ldots < a_{d_k}$$

and thus also

$$x_{F(d_1)} < x_{F(d_2)} < \dots < x_{F(d_k)}$$

Case 1: $|D|$ odd. Let $\ell \in \mathbb{N} \setminus D$ be minimal and consider the place of a_ℓ in the ordering

$$a_{d_1} < \dots < a_{d_i} < a_\ell < a_{d_{i+1}} < \dots < a_{d_k} .$$

Then since $(X,<)$ is a dense linear ordering w/o endpoints, there is some $x_j \in X$ with the same relative position:

$$x_{F(d_1)} < \dots < x_{F(d_i)} < x_j < x_{F(d_{i+1})} < \dots < x_{F(d_k)} .$$

Now, let $F(\ell) = j$. Then clearly $(*)$ holds for $m, n \in D \cup \{\ell\}$.

Case 2 $|D|$ even. Let $j \in \mathbb{N} \setminus F[D]$ be minimal and consider x_j's place in the ordering:

$$x_{F(d_1)} < \dots < x_{F(d_i)} < x_j < x_{F(d_{i+1})} < \dots < x_{F(d_k)}$$

Again, find a_ℓ s.t. $a_{d_i} < a_\ell < a_{d_{i+1}}$ and let $F(\ell) = j$.

Thus F will be a bijection and (*) holds.

$\square$